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# The Generalized Integer Gamma Distribution—A Basis for Distributions in Multivariate Statistics

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In this paper the distribution of the sum of independent $\Gamma$ variables with different scale parameters is obtained by direct integration, without involving a series expansion. One of its particular cases is the distribution of the product of some particular independent Beta variables. Both distributions are obtained in a concise manageable form readily applicable to research work in the area of multivariate statistics distributions. The exact distribution of the generalized Wilks' $\Lambda$ statistic is then obtained as a direct application of the results. © 1998 Academic Press

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## 1. INTRODUCTION

The distribution of the sum of independent Gamma random variables with different scale parameters and the distribution of the product of particular independent Beta random variables has been studied by a few authors. All the results have been obtained under the form of series expansions. Indeed, Kabe [7], by inverting the characteristic function of a linear combination with all positive coefficients of independent Gamma variates, obtained its distribution in terms of a hypergeometric series. Tretter and Walster [10] and Nandi [9] expressed the distribution of the product of particular independent Beta variates as a mixture of incomplete Beta distributions, involving a series representation. Gupta and Richards [6] by expanding the exponential term and using term by term integration presented the distribution of the sum of independent Gamma variables with different parameters also as a series expansion.

Let

$$Y_i \sim \Gamma(r_i, \lambda_i), \qquad r_i, \lambda_i > 0$$



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be a short notation for the fact that the probability density function (p.d.f.) of $Y_i$ is a Gamma distribution with shape parameter $r_i$ and scale parameter $\lambda_i$, this is

$$f_{Y_i}(y_i)=\frac{\lambda_i^{r_i}}{\Gamma(r_i)}\, y_i^{r_i-1}e^{-\lambda_i y_i}, \qquad y_i>0, \quad r_i, \lambda_i>0. \tag{1}$$

Now let $Y_1, ..., Y_n$ be independent random variables having distributions given by (1), with $\lambda_1=\cdots=\lambda_n=\lambda$, and let $Y=Y_1+\cdots+Y_n$. Then it is a well known fact that

$$Y\sim\Gamma(r, \lambda)$$

with $r=r_1+\cdots+r_n$.

In this paper we will obtain the distribution of $Y=Y_1+\cdots+Y_n$ under the situation where all the $\lambda_i$'s are different, without involving a series expansion, as long as all the $r_i$'s are integer. We will call such distribution a Generalized Integer (GI) Gamma distribution and after some simplifications it is presented in a concise and easily manageable form. Not only is this distribution a generalization of the Gamma distribution, but also one of its particular cases is the distribution of the logarithm of the product of an even number of particular independent Beta random variables. Therefore such particular GI Gamma distribution may be used to obtain the distributions of a number of multivariate statistics.

## 2. A PRELIMINARY RESULT

A key result for the developments ahead is the following.

*Result* 1. For integer $p$ and $q$, and for $z>0$,

$$\begin{aligned}&\int_0^z (z-x)^{q-1}\, x^{p-1}e^{-kx}\, dx\\ &\quad=(-1)^q\,(q-1)!\left(\sum_{j=0}^{p-1}\frac{(p-1)!}{j!}\,a_{p-j,\,q}z^jk^{j-q-p+1}\right)e^{-kz}\\ &\qquad+(p-1)!\sum_{j=0}^{q-1}(-1)^{q-1-j}\,\frac{(q-1)!}{j!}\,a_{q-j,\,p}z^jk^{j-q-p+1}\end{aligned}$$

$$= (p-1)!\,(q-1)!\left\{(-1)^q\left(\sum_{j=1}^{p}\frac{a_{p-j+1,\,q}}{(j-1)!}\,z^{j-1}k^{j-q-p}\right)e^{-kz}\right.$$
$$\left.+(-1)^p\left(\sum_{j=1}^{q}\frac{a_{q-j+1,\,p}}{(j-1)!}\,z^{j-1}(-k)^{j-p-q}\right)\right\} \qquad (\text{for } k\neq 0)$$
$$\left(=\frac{z^{p+q-1}}{a_{p,\,q}(p+q-1)}\text{ for } k=0\right)$$

where

$$a_{j,\,p}=\sum_{i_p=1}^{j}\sum_{i_{p-1}=1}^{i_p}\cdots\sum_{i_2=1}^{i_3}\sum_{i_1=1}^{i_2}1=\sum_{i=1}^{j}a_{i,\,p-1}=a_{p,\,j}$$
$$=\binom{j+p-2}{p-1}=\binom{j+p-2}{j-1}, \qquad j,\,p\geqslant 1, \tag{2}$$

with

$$a_{i,\,1}=a_{1,\,i}=1,$$

for all $i\geqslant 1$.

The above result is obtained on integrating successively by parts.

## 3. TWO USEFUL DISTRIBUTIONS

Using Result 1 we may easily obtain the distribution of the sum of independent Gamma random variables with different scale parameters and integer shape parameters.

We have then the following theorem.

THEOREM 1. *Let*

$$Y_i\sim\Gamma(r_i,\lambda_i) \qquad i=1,2,$$

*be two independent Gamma random variables.*

*Then, if* $r_i (i=1,2)$ *are integer, the p.d.f. of* $Z=Y_1+Y_2$ *is, for* $z>0$,

$$f_Z(z)=\left(\prod_{i=1}^{2}\lambda_i^{r_i}\right)\sum_{i=1}^{2}(-1)^{S-r_i}\left(\sum_{j=1}^{r_i}\frac{a_{r_i-j+1,\,S-r_i}}{(j-1)!}\,z^{j-1}(2\lambda_i-L)^{j-S}\right)e^{-\lambda_i z},$$
$$(\text{for } \lambda_1\neq\lambda_2)$$
$$\left(=\frac{\lambda^S}{\Gamma(S)}\,z^{S-1}e^{-\lambda z}\text{ for } \lambda_1=\lambda_2=\lambda\right), \tag{3}$$

*where*

$$S = r_1 + r_2,\ L = \lambda_1 + \lambda_2$$

*and the* $a_{r_i - j+1,\, S-r_i}$ *are defined as in* (2).

*Proof.* From (1), the distribution of $Z = Y_1 + Y_2$, for $\lambda_1 \neq \lambda_2$, taking into account the independence of $Y_1$ and $Y_2$, and Result 1, is

$$\begin{aligned} f_Z(z) &= \int_0^z \frac{\lambda_1^{r_1} \lambda_2^{r_2}}{\Gamma(r_1)\, \Gamma(r_2)}\, y_1^{r_1 - 1} e^{-\lambda_1 y_1} (z - y_1)^{r_2 - 1}\, e^{-\lambda_2 (z - y_1)}\, dy_1 \\ &= \frac{\lambda_1^{r_1} \lambda_2^{r_2} e^{-\lambda_2 z}}{(r_1 - 1)!\, (r_2 - 1)!} \int_0^z (z - y_1)^{r_2 - 1}\, y_1^{r_1 - 1} e^{-(\lambda_1 - \lambda_2)\, y_1}\, dy_1 \\ &= \lambda_1^{r_1} \lambda_2^{r_2} \left\{ (-1)^{r_2} \left( \sum_{j=1}^{r_1} \frac{a_{r_1 - j+1,\, r_2}}{(j-1)!}\, z^{j-1} (\lambda_1 - \lambda_2)^{j - r_1 - r_2} \right) e^{-\lambda_1 z} \right. \\ &\quad \left. + (-1)^{r_1} \left( \sum_{j=1}^{r_2} \frac{a_{r_2 - j+1,\, r_1}}{(j-1)!}\, z^{j-1} (\lambda_2 - \lambda_1)^{j - r_2 - r_1} \right) e^{-\lambda_2 z} \right\}. \end{aligned}$$

COROLLARY 1. *Let*

$$Y_i \sim \Gamma(r_i, \lambda_i), \qquad i = 1, ..., g \geqslant 2,$$

*be g independent Gamma random variables.*

*Then, if* $r_i$ $(i = 1, ..., g)$ *are integer, the p.d.f. of* $Z = Y_1 + \cdots + Y_g$ *is, for* $z > 0$,

$$\begin{aligned} f_Z(z) = K_i^g \sum_{i=1}^{g} \Bigg( S_i \sum_{j_1 = 1}^{r_i} a_{r_i - j_1 + 1,\, r_1^{*i}} (\lambda_i - \lambda_1^{*i})^{j_1 - r_i - r_1^{*i}} \\ \sum_{j_2 = 1}^{j_1} a_{j_1 - j_2 + 1,\, r_2^{*i}} (\lambda_i - \lambda_2^{*i})^{j_2 - j_1 - r_2^{*i}} \\ \cdots \sum_{j_{g-1} = 1}^{j_{g-2}} \frac{a_{j_{g-2} - j_{g-1} + 1,\, r_{g-1}^{*i}}}{(j_{g-1} - 1)!} (\lambda_i - \lambda_{g-1}^{*i})^{j_{g-1} - j_{g-2} - r_{g-1}^{*i}}\, z^{j_{g-1} - 1} \Bigg) e^{-\lambda_i z} \end{aligned}$$

$$(\text{for } \lambda_i \neq \lambda_{i'},\ i, i' \in \{1, ..., g\},\ i \neq i')$$

$$\left( = \frac{\lambda^S}{\Gamma(S)}\, z^{S-1} e^{-\lambda z} \text{ for } \lambda_1 = \cdots = \lambda_g = \lambda \right) \tag{4}$$

*where* $f_Z(z)$ *is* $f_Z(z; \lambda_1, ..., \lambda_g; r_1, ..., r_g)$, *the coefficients* $a_{j,\, r^*}$ *are given by* (2),

$$K_i^g = \prod_{i=1}^{g} \lambda_i^{r_i}, \qquad S_i = (-1)^{S - r_i}, \qquad S = \sum_{i=1}^{g} r_i, \tag{5}$$

*$\lambda_j^{*i}$ is the jth element of the set $\{\lambda_1, ..., \lambda_g\} \backslash \{\lambda_i\}$ and similarly $r_j^{*i}$ is the jth element of the set $\{r_1, ..., r_g\} \backslash \{r_i\}$, where "\" denotes set difference. Equivalently $\lambda_j^{*i}$ and $r_j^{*i}$ may be defined as*

$$\lambda_j^{*i} = \begin{cases} \lambda_j & i > j \\ \lambda_{j+1} & i \leqslant j \end{cases} \qquad r_j^{*i} = \begin{cases} r_j & i > j \\ r_{j+1} & i \leqslant j \end{cases} \qquad i = 1, ..., g; \quad j = 1, ..., g-1.$$

*Proof.* Theorem 1 enables us to obtain the distribution of $Z_1 = Y_1 + Y_2$. Then applying Result 1 to the joint distribution of $Z_1$ and $Y_3$ we get the distribution of $Z_2 = Z_1 + Y_3 = Y_1 + Y_2 + Y_3$, and so on. For $\lambda_1 = \cdots = \lambda_g = \lambda$ the result is known. ∎

Distributions (3) and (4) may be seen as generalizations of the common Gamma distribution. Indeed, if $\lambda_1 = \lambda_2$ in (3) or $\lambda_1 = \cdots = \lambda_g$ in (4) then we have a common Gamma distribution and if $r_1 = r_2 = 1$ in (3) or $r_1 = \cdots = r_g = 1$ in (4) we have the sum of independent exponentials. So we may call the distributions (3) and (4) as Generalized Integer (GI) Gamma distributions.

As a matter of fact further simple changes in the summation order yield distribution (4) under the form

$$f_Z(z) = K_i^g \sum_{i=1}^{g} P_i(z)\, e^{-\lambda_i z} \qquad (z > 0) \tag{6}$$

where $P_i(z)$ is a polynomial of degree $r_i - 1$ in $z$ which may be written as

$$P_i(z) = \sum_{k=1}^{r_i} c_{i,k}(g, \underline{r}, \underline{\lambda})\, z^{k-1} \tag{7}$$

where

$$c_{i,k}(g, \underline{r}, \underline{\lambda}) = S_i \sum_{j_1=k}^{r_i} \sum_{j_2=k}^{j_1} \cdots \sum_{j_{g-2}=k}^{j_{g-3}} \alpha_{j_1} \beta_{j_2} \cdots \gamma_{j_{g-2}} \delta_k \tag{8}$$

with

$$\begin{aligned} \underline{r} &= (r_1, ..., r_g)' \\ \underline{\lambda} &= (\lambda_1, ..., \lambda_g)' \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
\alpha_{j_1} &= a_{r_i - j_1 + 1,\, r_1^{*i}} (\lambda_i - \lambda_1^{*i})^{j_1 - r_i - r_1^{*1}} \\
\beta_{j_2} &= a_{j_1 - j_2 + 1,\, r_2^{*i}} (\lambda_i - \lambda_2^{*i})^{j_2 - j_1 - r_2^{*i}} \\
\gamma_{j_{g-2}} &= a_{j_{g-3} - j_{g-2} + 1,\, r_{g-2}^{*i}} (\lambda_i - \lambda_{g-2}^{*i})^{j_{g-2} - j_{g-3} - r_{g-2}^{*i}} \\
\delta_k &= \frac{a_{j_{g-2} - k + 1,\, r_{g-1}^{*i}}}{(k-1)!} (\lambda_i - \lambda_{g-1}^{*i})^{k - j_{g-2} - r_{g-1}^{*i}}.
\end{aligned}
\tag{10}
$$

Expression (6) is easily obtained from (4) by simply interchanging the summations. We may note that, using the notations in (10) and $k$ for $j_{g-1}$, distribution (4) may be written as

$$
\begin{aligned}
f_Z(z) &= K_i^g \sum_{i=1}^{g} \left( S_i \sum_{j_1=1}^{r_i} \alpha_{j_1} \sum_{j_2=1}^{j_1} \beta_{j_2} \cdots \sum_{j_{g-2}=1}^{j_{g-3}} \gamma_{j_{g-2}} \sum_{k=1}^{j_{g-2}} \delta_k z^{k-1} \right) e^{-\lambda_i z} \\
&= K_i^g \sum_{i=1}^{g} \left( S_i \sum_{j_1=1}^{r_i} \sum_{j_2=1}^{j_1} \cdots \sum_{j_{g-2}=1}^{j_{g-3}} \sum_{k=1}^{j_{g-2}} \alpha_{j_1} \beta_{j_2} \cdots \gamma_{j_{g-2}} \delta_k z^{k-1} \right) e^{-\lambda_i z} \\
&= K_i^g \sum_{i=1}^{g} \left\{ \underbrace{\sum_{k=1}^{r_i} \left( S_i \sum_{j_1=k}^{r_i} \sum_{j_2=k}^{j_1} \cdots \sum_{j_{g-2}=k}^{j_{g-3}} \alpha_{j_1} \beta_{j_2} \cdots \gamma_{j_{g-2}} \delta_k \right)}_{c_{i,k}} z^{k-1} \right\} e^{-\lambda_i z}.
\end{aligned}
$$

Expression (8) is quite easy to compute but may become a bit long even for moderately large values of $r_i$, given the nesting of the summations. However, a little algebra will allow us to get a much faster and easier way to compute the coefficients $c_{i,k}(g, \underline{r}, \underline{\lambda})(k = 1, ..., r_i)$, so that in many cases the coefficients may even be computed by hand.

From (8) we can see that, for a given $i$, the easiest coefficient to compute is the one associated with the highest degree in the polynomial, which may be given by

$$
c_{i, r_i}(g, \underline{r}, \underline{\lambda}) = \frac{1}{(r_i - 1)!} \prod_{\substack{j=1 \\ j \neq i}}^{g} (\lambda_j - \lambda_i)^{-r_j}. \tag{11}
$$

Then, for $k = 1, ..., r_i - 1$, we have

$$
\begin{aligned}
&c_{i, r_i - k}(g, \underline{r}, \underline{\lambda}) \\
&\quad = \frac{1}{k} \sum_{j=1}^{k} \frac{(r_i - k + j - 1)!}{(r_i - k - 1)!} R(j-1, i, g, \underline{r}, \underline{\lambda})\, c_{i, r_i - (k-j)}(g, \underline{r}, \underline{\lambda}),
\end{aligned}
\tag{12}
$$

where

$$R(n,\, j,\, g,\, \underline{r},\, \underline{\lambda}) = \sum_{\substack{i=1\\ i\neq j}}^{g} r_i(\lambda_j - \lambda_i)^{-n-1}, \qquad (n=0,\, ...,\, r_i-1). \tag{13}$$

Using the above notation it is then very easy to get the cumulative distribution function (c.d.f.) of $Z$. For non-negative integer $j$ and real $\lambda$, we obtain

$$\int_0^w z^j e^{-\lambda z} dz = \frac{j!}{\lambda^{j+1}} \left\{ 1 - \left( \sum_{i=0}^{j} \frac{\lambda^i w^i}{i!} \right) e^{-\lambda w} \right\}, \qquad w>0, \tag{14}$$

on integrating by parts. Then, from (6), (7) and (14), the c.d.f. of $Z$ is

$$F_Z(z) = K_i^g \sum_{i=1}^{g} P_i^*(z) \qquad (z>0) \tag{15}$$

where $P_i^*(z)$ is a polynomial of degree $r_i - 1$ in $z$, with

$$P_i^*(z) = \sum_{k=1}^{r_i} c_{i,k}(g,\, \underline{r},\, \underline{\lambda})\, \frac{(k-1)!}{\lambda_i^k} \left\{ 1 - \left( \sum_{j=0}^{k-1} \frac{\lambda_i^j z^j}{j!} \right) e^{-\lambda_i z} \right\}. \tag{16}$$

Expressions (15) and (16) show that $F_Z(z)$ may be seen as a generalization of the Incomplete Gamma function, as one would expect.

Indeed, the expression for the c.d.f. of $Z$ may be further simplified since from (15) and (16)

$$\begin{aligned} F_Z(z) = {} & K_i^g \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}(g,\, \underline{r},\, \underline{\lambda})\, \frac{(k-1)!}{\lambda_i^k} \\ & - K_i^g \sum_{i=1}^{g} e^{-\lambda_i z} \sum_{k=1}^{r_i} c_{i,k}(g,\, \underline{r},\, \underline{\lambda})(k-1)! \sum_{j=0}^{k-1} \frac{z^j}{j!\, \lambda_i^{k-j}} \end{aligned}$$

where, after some algebraic manipulation we may show that

$$\sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}(g,\, \underline{r},\, \underline{\lambda})\, \frac{(k-1)!}{\lambda_i^k} = \prod_{i=1}^{g} \lambda_i^{-r_i} = (K_i^g)^{-1}.$$

Thus $F_Z(z)$ may be further written as

$$\begin{aligned} F_Z(z) &= 1 - K_i^g \sum_{i=1}^{g} e^{-\lambda_i z} \sum_{k=1}^{r_i} c_{i,k}(g,\, \underline{r},\, \underline{\lambda})(k-1)! \sum_{j=0}^{k-1} \frac{1}{j!} \frac{z^j}{\lambda_i^{k-j}} \\ &= 1 - K_i^g \sum_{i=1}^{g} P_i^{**}(z)\, e^{-\lambda_i z} \qquad (z>0) \end{aligned} \tag{17}$$

where

$$P_i^{**}(z) = \sum_{k=1}^{r_i} c_{i,k}(g, \underline{r}, \underline{\lambda})(k-1)! \sum_{j=0}^{k-1} \frac{1}{j!} \frac{z^j}{\lambda_i^{k-j}} \tag{18}$$

is a polynomial of degree $r_i - 1$ in $z$.

Now $F_Z(z)$ is clearly shown to be a generalization of the Incomplete Gamma Function. In some cases, as we will see in Theorem 2, the distribution of $U = e^{-Z}$ may be also sought. From the p.d.f. of $Z$ we can readily obtain the p.d.f. of $U$, through the transformation of variable $Z = -\log(U)$. Since the Jacobian of the transformation is $1/u$, then

$$f_U(u) = K_i^g \sum_{i=1}^{g} P_i(-\log u)\, u^{\lambda_i - 1} \qquad (0 < u < 1) \tag{19}$$

where $P_i(-\log u)$, still given by (7), is now a polynomial of degree $r_i - 1$ in $(-\log u)$ with coefficients $c_{i,k}$.

Then the c.d.f. of $U$ is given by

$$F_U(u) = K_i^g \sum_{i=1}^{g} P_i^{**}(-\log u)\, u^{\lambda_i} \qquad (0 < u < 1) \tag{20}$$

where $P_i^{**}(-\log u)$, a polynomial of degree $r_i - 1$ in $(-\log u)$, is given by (18).

The above expression for $F_U(u)$ may be obtained either from (19) and (18), using, for non-negative integer $j$ and real $\lambda$,

$$\int_0^w (-\log l)^j\, l^\lambda\, dl = w^{\lambda+1} j! \sum_{i=0}^{j} \frac{1}{i!} \frac{(-\log w)^i}{(\lambda+1)^{j-i+1}}, \qquad 0 < w < 1$$

which is easily obtained on integrating by parts or, equivalently, through the change of variable $z = -\log t$

$$\begin{aligned} F_U(u) &= \int_0^u f_U(t)\, dt = \int_0^u f_{-\log U}(-\log t) \frac{1}{t}\, dt = \int_u^0 -f_{-\log U}(-\log t) \frac{1}{t}\, dt \\ &= \int_{-\log u}^{+\infty} f_Z(z)\, dz = \int_0^{+\infty} f_Z(z)\, dz - \int_0^{-\log u} f_Z(z)\, dz \\ &= 1 - F_Z(-\log u). \end{aligned} \tag{21}$$

Expression (21) clearly shows that it is indeed equivalent to use $Z = -\log U$ or $U = e^{-Z}$ to carry out tests. For example, using (21) above, we see that the $\alpha$-percentile of $U$ is equal to the exponential of the symmetrical of the $(1-\alpha)$-percentile of $Z$, and conversely, the $\alpha$-percentile of $Z$ is equal to the symmetrical of the logarithm of the $(1-\alpha)$-percentile of $U$.

A particular and important GI Gamma distribution arises in the following situation.

THEOREM 2. *Let $p = 2m$ be an even integer, with $m \geqslant 1$, and*

$$Y_j \sim B\left(a_j, \frac{b}{2}\right) \qquad j = 1, ..., p$$

*be independent random variables with Beta distributions, where b is a positive integer and $a_j = k - j/2 (j = 1, ..., p)$, with $k > p/2$. Further let*

$$W' = \prod_{j=1}^{p} Y_j$$

*and*

$$W = -\log W' = -\sum_{j=1}^{p} \log Y_j.$$

*Then W is the sum of $p + b - 2$ independent Gamma distributions with parameters $\lambda_j = k + (j - p - 1)/2$ $(j = 1, ..., p + b - 2)$ and integer parameters $r_j$ given by*

$$r_j = \begin{cases} h_j & j = 1, 2 \\ r_{j-2} + h_j & j = 3, ..., p + b - 2 \end{cases} \tag{22}$$

*where*

$$h_j = (\text{number of elements of } \{p, b\} \text{ greater or equal to } j) - 1. \tag{23}$$

*Thus the p.d.f. and c.d.f. of W are*

$$f_W(w) = K_j^{p+b-2} \sum_{j=1}^{p+b-2} P_j(w)\, e^{-\lambda_j w} \qquad (w > 0) \tag{24}$$

*and*

$$F_W(w) = 1 - K_j^{p+b-2} \sum_{j=1}^{p+b-2} P_j^{**}(w)\, e^{-\lambda_j w} \qquad (w > 0), \tag{25}$$

*with $K_j^{p+b-2}$ given by (5) and $P_j$ and $P_j^{**}$ given by (7) and (18) respectively.*

*The p.d.f. and c.d.f. of W′ are thus*

$$f_{W'}(w) = K_j^{p+b-2} \sum_{j=1}^{p+b-2} P_j(-\log w)\, w^{\lambda_j - 1} \qquad (0 < w < 1) \tag{26}$$

*and*

$$F_{W'}(w) = K_j^{p+b-2} \sum_{j=1}^{p+b-2} P_j^{**}(-\log w)\, w^{\lambda_j} \qquad (0 < w < 1). \tag{27}$$

*Here, the coefficients* $c_{j,k}$ *in the definitions of* $P_j$ *and* $P_j^{**}$ *are given by*

$$\begin{aligned} c_{j,r_j}(g, \underline{r}) &= \frac{2^{S-r_j}}{(r_j-1)!\, \prod_{i=1,\, i\neq j}^{g} (i-j)^{r_i}} \\ &= \frac{(-1)^{r_{j+1}}\, 2^{S-r_j}}{(r_j-1)!\, \prod_{i=1}^{j-2} (i-j)^{r_i} \prod_{i=j+2}^{g} (i-j)^{r_i}} \end{aligned} \tag{28}$$

*and, for* $k = 1, ..., r_j - 1$,

$$c_{j,r_j-k}(g, \underline{r}) = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!} R(i-1, j, g, \underline{r})\, c_{j,r_j-(k-i)}(g, \underline{r}), \tag{29}$$

*where*

$$R(n, i, g, \underline{r}) = \sum_{\substack{j=1 \\ j \neq i}}^{g} r_j \left(\frac{2}{i-j}\right)^{n+1} \qquad (n = 0, ..., r_j - 1), \tag{30}$$

*with* $\underline{r}$ *as in* (9).

*In* (28), $S = r_1 + \cdots + r_g = pb/2$ *is the number of exponentials that incorporate the distribution of* $W$.

*Proof.* Given the independence of the $p$ Beta random variables, the $h$th moment of $W'$ is

$$E(W'^h) = \prod_{j=1}^{p} \frac{\Gamma(a_j + h)\, \Gamma(a_j + (b/2))}{\Gamma(a_j + (b/2) + h)\, \Gamma(a_j)}, \tag{31}$$

and thus the characteristic function of $W$ is

$$E(e^{itW}) = E(e^{-it \log W'}) = E(W'^{-it}) = \prod_{j=1}^{p} \frac{\Gamma(a_j - it)\, \Gamma(a_j + (b/2))}{\Gamma(a_j + (b/2) - it)\, \Gamma(a_j)},$$

where $i = (-1)^{1/2}$ and $t$ is a real constant. But then, given that $p = 2m$ is an even integer, using the duplication formula for the Gamma function

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\, \Gamma(z + \tfrac{1}{2})$$

we may write,

$$\begin{aligned}\prod_{j=1}^{p} \Gamma(a_j - it) &= \prod_{j=1}^{m} \Gamma\left(k - \frac{2j-1}{2} - it\right) \Gamma\left(k - \frac{2j}{2} - it\right) \\ &= \prod_{j=1}^{m} \Gamma\left(k - \frac{2j}{2} + \frac{1}{2} - it\right) \Gamma\left(k - \frac{2j}{2} - it\right) \\ &= \pi^{m/2} \prod_{j=1}^{m} \frac{1}{2^{2(k-j)-2it-1}} \Gamma(2k - 2j - 2it)\end{aligned}$$

and similarly

$$\prod_{j=1}^{p} \Gamma\left(a_j + \frac{b}{2}\right) = \pi^{m/2} \prod_{j=1}^{m} \frac{1}{2^{2(k-j)+b-1}} \Gamma(2k - 2j + b)$$

$$\prod_{j=1}^{p} \Gamma\left(a_j + \frac{b}{2} - it\right) = \pi^{m/2} \prod_{j=1}^{m} \frac{1}{2^{2(k-j)+b-2it-1}} \Gamma(2k - 2j + b - 2it)$$

$$\prod_{j=1}^{p} \Gamma(a_j) = \pi^{m/2} \prod_{j=1}^{m} \frac{1}{2^{2(k-j)-1}} \Gamma(2k - 2j).$$

Then

$$\begin{aligned}E(e^{itW}) &= \prod_{j=1}^{m} \frac{2^{2(k-j)+b-2it-1} 2^{2(k-j)-1}}{2^{2(k-j)-2it-1} 2^{2(k-j)+b-1}} \frac{\Gamma(2k-2j-2it)\, \Gamma(2k-2j+b)}{\Gamma(2k-2j+b-2it)\, \Gamma(2k-2j)} \\ &= \prod_{j=1}^{m} \frac{\Gamma(2k-2j+b)/\Gamma(2k-2j)}{\Gamma(2k-2j+b-2it)/\Gamma(2k-2j-2it)}\end{aligned}$$

which, using for all real or complex $a$ and integer $n$,

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{i=0}^{n-1} (a+i),$$

gives

$$\begin{aligned}E(e^{itW}) &= \prod_{j=1}^{m} \prod_{i=0}^{b-1} (2k-2j+i)\, (2k-2j+i-2it)^{-1} \\ &= \prod_{j=1}^{m} \prod_{i=0}^{b-1} \left(\frac{2k-2j+i}{2}\right)\left(\frac{2k-2j+i}{2} - it\right)^{-1} \\ &= \prod_{j=1}^{2m+b-2} \left(k + \frac{j-2m-1}{2}\right)^{r_j} \left(k + \frac{j-2m-1}{2} - it\right)^{-r_j} \qquad (32)\end{aligned}$$

where $r_j$ are integers given by (22), with $h_j$ defined as

$$h_j = \begin{cases} 1 & j = 1, ..., \min(2m, b) \\ 0 & j = 1 + \min(2m, b), ..., \max(2m, b), \\ -1 & j = 1 + \max(2m, b), ..., 2m + b - 2 \end{cases}$$

or equivalently defined as in (23).

Expression (32) shows that $W$ is the sum of $S = mb = pb/2$ independent exponentials with parameters $k - j + i/2$ ($j = 1, ..., m$; $i = 0, ..., b$) or the sum of $2m + b - 2 = p + b - 2$ independent Gamma random variables with parameters $r_j$ and $\lambda_j = k + (j - p - 1)/2$ ($j = 1, ..., p + b - 2$).

This way the p.d.f. of $W$ is a GI Gamma distribution, as defined in (4), with

$$g = p + b - 2$$

$$\lambda_j = k + \frac{j - 2m - 1}{2}.$$

Thus, in order to obtain the p.d.f. and c.d.f. of $W$ in (24) and (25) we just have to replace $g$ by $p + b - 2$ and $Z$ by $W$ in (6) and (17). Similarly, the p.d.f. and c.d.f. of $W'$ in (26) and (27) are obtained by replacing $g$ by $p + b - 2$ and $U$ by $W'$ in (19) and (20).

The coefficients $c_{j,k}$, $k = 1, ..., r_j$; $j = 1, ..., g$, in the polynomials $P_j$ and $P_j^{**}$ are given by (11) through (13), which in this particular case, given that $\lambda_i - \lambda_j = (i - j)/2$, may be written as in (28) through (30). ∎

We may notice that in Theorem 2, $p$ and $b$ are interchangeable as long as they are both even, and that when $b = 1$ the distribution in (26) becomes a Beta distribution.

Theorem 2 may be seen as an extension of the well known result

$$-n \log\{X/(X + Y)\} \sim \chi_2^2$$

(Fujikoshi and Mukaihata [5]), for two independent random variables $X \sim \chi_n^2$ and $Y \sim \chi_2^2$.

## 4. APPLICATION TO THE EXACT DISTRIBUTION OF THE (GENERALIZED) WILKS' $\Lambda$

An useful and interesting application of the GI Gamma distribution is in obtaining the distribution of some multivariate statistics. The Wilks' $\Lambda$ statistic is among such statistics.

The Wilks' Lambda (Wilks [11, 12]) is a well known statistic used, under a multivariate normal setting, to test in multivariate analysis of variance the existence of overall differences among the level means of a factor, in multivariate regression or canonical analysis to test the equality of the vector of regression parameters or part of it to a given vector or still used to test the independence of $m$ sets of normally distributed variables (Wilks [11, 12]; Bartlett [2]; Anderson [1, Chap. 8, 9]; Kshirsagar [8, Chap. 8]). Since all the above settings lead to the same test statistic, in order to keep it short we will describe this statistic only under the last setting.

Let $\underset{\sim}{x}$ be a $p \times 1$ vector of variables with a joint $p$-multivariate normal distribution $N_p(\underset{\sim}{\mu}, \Sigma)$. Further let $\underset{\sim}{x}$ be split into $m$ subvectors, the $k$th of which has $p_k$ variables, with $p = p_1 + \cdots + p_m$. Then each subvector $\underset{\sim}{x}_k (k = 1, ..., m)$ has a $p_k$-multivariate normal distribution $N_{p_k}(\underset{\sim}{\mu}_k, \Sigma_{kk})$ and, for a sample of size $n$, the $(2/n)$th power of the likelihood ratio statistic to test the null hypothesis

$$H_0 \colon \Sigma = \operatorname{diag}(\Sigma_{11}, ..., \Sigma_{kk}, ..., \Sigma_{mm}), \tag{33}$$

i.e., the hypothesis of independence of the $m$ subvectors $\underset{\sim}{x}_k$, is the Wilks' Lambda,

$$\Lambda = \frac{|A|}{\prod_{k=1}^{m} |A_{kk}|}, \tag{34}$$

where $|\cdot|$ stands for the determinant and

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} & \cdots & A_{km} \\ \vdots & & \vdots & & \vdots \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{mm} \end{bmatrix}$$

is either the sample variance-covariance matrix of the $p$ variables in $\underset{\sim}{x}$ or the Maximum Likelihood Estimator of $\Sigma$.

For $m > 2$ such a test statistic also arises under the setting of the Generalized Canonical Analysis, when the approach proposed by Carroll [3] is considered (Coelho, [4, Chap. 4]).

A general concise expression for the exact distribution of the (generalized) Wilks' Lambda, without involving any series expansion or unknown coefficients, may then be obtained by direct application of the result in Theorem 2 as long as at most one of the sets of variables has an odd number of variables.

### 4.1. *The General Case of $m \geqslant 2$ Sets of Variables*

For general $m \geqslant 2$ the Wilks' $\Lambda$ statistic in (34) to test the independence of the $m$ sets of variables $\underline{x}_1, ..., \underline{x}_m$, may be written as

$$\Lambda = \prod_{k=1}^{m-1} \Lambda_{k(k+1, ..., m)} \tag{35}$$

where $\Lambda_{k(k+1, ..., m)}$ stands for the Wilks' $\Lambda$ statistic to test the independence of the set of variables $\underline{x}_k$ and the set formed by the joining of the sets $(\underline{x}_{k+1}, ..., \underline{x}_m)$ (Anderson [1, Theorem 9.3.2]; Coelho [4, Sec. 4.7]).

Under the hypothesis of joint multivariate normality of the $m$ sets of variables and the null hypothesis (33), of independence of the $m$ sets of variables, the $m-1$ Wilks' Lambda statistics $\Lambda_{k(k+1, ..., m)}$ are all independent (Coelho [4, Chap. 4]). Considering that $\underline{x}_k$ has $p_k$ variables ($k = 1, ..., m$), the distribution of $\Lambda_{k(k+1, ..., m)}$ is the same as the distribution of $\prod_{j=1}^{p_k} Y_j$ where, for a sample of size $n+1$, with $n \geqslant p_1 + \cdots + p_m$, $Y_j$ are $p_k$ independent Beta random variables with $Y_j \sim B((n+1-q_k-j)/2, q_k/2)$, using $q_k = p_{k+1} + \cdots + p_m$ (Anderson [1, Theorem 9.3.2]). Then we have the following Theorem.

THEOREM 3. *For the general case of $m \geqslant 2$ sets of variables, the kth set having $p_k$ variables $(k = 1, ..., m)$, when at most one of the sets has an odd number of variables, and for a sample of size $n+1 > p_1 + \cdots + p_m$, the p.d.f. and c.d.f. of $W = -\log \Lambda$, under the null hypothesis* (33) *are given by*

$$f_W(w) = K_j^g \sum_{j=1}^{g} P_j(w)\, e^{-\lambda_j w}$$

*and*

$$F_W(w) = 1 - K_j^g \sum_{j=1}^{g} P_j^{**}(w)\, e^{-\lambda_j w},$$

*respectively, with*

$$g = p - 2 \qquad \textit{where} \quad p = \sum_{k=1}^{m} p_k, \tag{36}$$

*being $K_j^g$ given by* (5) *and $P_j(w)$ and $P_j^{**}$ defined as in* (7) *and* (18).

*The p.d.f. and c.d.f. of $\Lambda$ are then given by*

$$f_\Lambda(l) = K_j^g \sum_{j=1}^{g} P_j(-\log l)\, l^{\lambda_j - 1}$$

*and*

$$F_\Lambda(l) = K_j^g \sum_{j=1}^{g} P_j^{**}(-\log l)\, l^{\lambda_j}.$$

*The scale parameters are* $\lambda_j = (n-p+j)/2$ $(j=1, ..., g)$, *with* $p$ *given by* (36) *above. The shape parameters* $r_j (j=1, ..., g)$, *are given by*

$$r_j = \begin{cases} h_j & j=1,2 \\ r_{j-2}+h_j & j=3, ..., g=p_1+\cdots+p_m-2, \end{cases}$$

*an extension of* (22), *with the* $h_j$*'s defined as*

$$h_j = (\textit{number of } p_k (k=1, ..., m) \geqslant j) - 1, \qquad j=1, ..., g=p-2.$$

*Proof.* Given the independence of the $m-1$ Wilks' Lambdas in (35) we have

$$E(\Lambda^h) = \prod_{k=1}^{m-1} E(\Lambda^h_{k(k+1, ..., m)}).$$

When at most one of the sets has an odd number of variables, without any loss of generality, we may suppose it to be the $m$th set. Then, from the proof of Theorem 2 in Section 3,

$$\begin{aligned} E(e^{-it\log\Lambda}) = E(\Lambda^{-it}) &= \prod_{k=1}^{m-1} \prod_{j=1}^{p_k+q_k-2} \left(\frac{n-p_k-q_k+j}{2}\right)^{r_{kj}} \\ &\quad\times \left(\frac{n-p_k-q_k+j}{2}-it\right)^{-r_{kj}} \\ &= \prod_{j=1}^{p-2} \left(\frac{n-p+j}{2}\right)^{r_j} \left(\frac{n-p+j}{2}-it\right)^{-r_j} \end{aligned}$$

where $q_k = p_{k+1}+\cdots+p_m$, and $r_{kj} (k=1, ..., m-1; j=1, ..., p_k+q_k-2)$ are defined by

$$r_{kj} = \begin{cases} h_{kj} & j=1,2 \\ r_{k,j-2}+h_{kj} & j=3, ..., p_k+q_k-2 \end{cases} \tag{37}$$

with

$$h_{kj} = (\text{number of elements in } \{p_k, q_k\} \text{ greater or equal to } j) - 1 \tag{38}$$

and

$$r_j = \sum_{k=1}^{m-1} r_{kj} \tag{39}$$

with $r_{kj} = 0$ if $j > p_k + q_k - 2$. Therefore $r_j$ may be defined by (22), where now

$$h_j = \sum_{k=1}^{m-1} h_{kj} = (\text{number of } p_k (k=1, ..., m) \geqslant j) - 1, \qquad j = 1, ..., p-2. \tag{40}$$

This shows that the distribution of $-\log \Lambda$ is, for $m \geqslant 2$, the sum of $p-2$ Gamma distributions with scale parameters $\lambda_j = (n - p + j)/2$ $(j = 1, ..., p-2)$ and shape parameters $r_j$. Thus applying Theorem 2 we obtain the results in this Theorem. ∎

We may note that for $m = 2$ we obtain the distribution of the usual Wilks' Lambda and thus of each of the $m-1$ Wilks' Lambdas $\Lambda_{k(k+1, ..., m)}$ $(k = 1, ..., m-1)$ on the right hand side of (35). Further, we may also note that the shape parameters $r_j$, in the distribution of the generalized Wilks' Lambda, are the ordered sum of the shape parameters in the distribution of the $m-1$ Wilks' Lambdas $\Lambda_{k(k+1, ..., m)} (k = 1, ..., m-1)$.

For example, for $m = 3$, with $p_1 = 4$, $p_2 = 4$ and $p_3 = 3$, we have $g = p_1 + p_2 + p_3 - 2 = 9$, and from (35) through (38),

$$\begin{aligned} \underline{h} &= [h_j] = [2, 2, 2, 1, -1, -1, -1, -1, -1]' \\ \underline{r} &= [r_j] = [2, 2, 4, 3, 3, 2, 2, 1, 1]' \\ \underline{r}_1 &= [r_{1j}] = [1, 1, 2, 2, 2, 2, 2, 1, 1]' \\ \underline{r}_2 &= [r_{2j}] = [1, 1, 2, 1, 1]', \end{aligned}$$

where the elements of $\underline{r}_1$ are the shape parameters in the distribution of $\Lambda_{1(2,3)}$, the Wilks' Lambda statistic used to test the independence between the first set and the superset formed by joining the second and third sets of variables, while in $\underline{r}_2$ are the shape parameters in the distribution of $\Lambda_{2(3)}$, the Wilks' Lambda statistic used to test the independence between the second and third sets of variables.

## 5. DISCUSSION

The GI Gamma distribution and the distributions presented in Theorem 2 of Section 3 are of major relevance in obtaining the distribution of several multivariate statistics whose moments are of the form in (31). Based on the above results it is possible to obtain the distributions of these statistics in a general, more concise and manageable form. This will make easier the computation of percentiles and enable us to overcome the problems arising from the use of distributions under a series expansion form.

As an illustration, the null distribution of the Wilks' $\Lambda$ statistic to test the independence of $m$ sets of variables is explicitly obtained in a simple manageable form even for general $m$.

Moreover, the use of the GI Gamma distribution also enables us to get a deeper insight and at the same time have an overall view upon the studies carried out so far on the distributions of a number of multivariate statistics, namely the Wilks' Lambda.

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